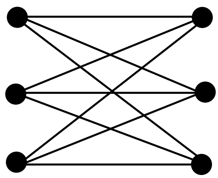
6. PLANAR GRAPHS

§6.1. Planar Graphs

A **planar graph** is one that can be drawn on a plane without edges crossing. Clearly the graph in example 5 is planar.

Example 1: The following is $\mathbf{K_{3,3}}$:

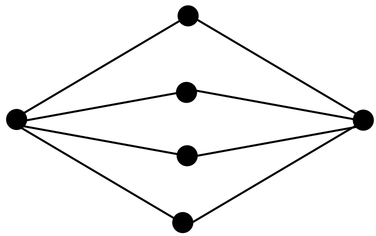


This graph was once featured in an Air New Zealand advertisement, where the 6 vertices consisted of the cities Brisbane, Sydney, Melbourne, Auckland, Wellington and Christchurch. The edges represented the trans-Tasman routes. This graph is not planar. You'll be easily convinced of this fact if you try to draw it without edges crossing, but that won't constitute a proof that it can't be done. However it can be drawn on a doughnut without edges crossing!

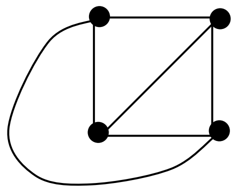
The question of whether certain graphs can be drawn on certain surfaces is an interesting question in Topology. Planarity is discussed in a later chapter. For other surfaces it is discussed in my notes on Topology.

§6.2. Euler's Formula

A graph is defined to be **planar** if it can be drawn on a plane so that edges do not cross. So $\mathbf{K}_{3,3}$ is not planar. But $\mathbf{K}_{4,2}$ is:

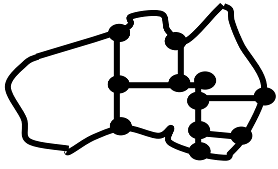


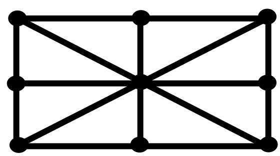
So is $\mathbf{K}_4$:



When a planar graph is drawn on a plane in such a way we call it a **map**. The regions enclosed by the edges, together with the outside region, are called **faces**. We often denote the numbers of vertices, faces and edges by V, F and E respectively.

Example 19: Some examples of maps, together with the numbers of vertices, faces and edges.





$$V = 11 \quad F = 6 \quad E = 16 \qquad\qquad V = 9 \quad F = 8 \quad E = 16$$

Note that in each case V + F – E = 1. This is always the case for planar graphs.

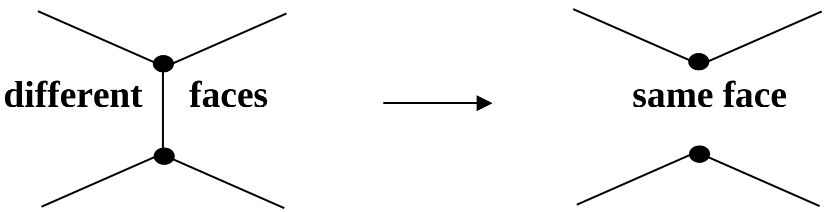
Theorem 3: (EULER'S FORMULA ON A PLANE)

For any connected planar map V + F – E = 1.

Proof: Define the Euler characteristic of a map to be

$$\chi = V + F - E.$$

If we remove an edge from a map that separates two faces and combine those two faces the value of χ will be unchanged.

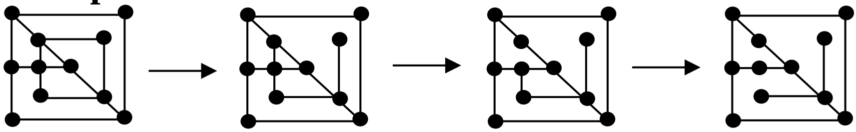


$$\left.\begin{array}{l} V \to V \\ F \to F - 1 \\ E \to E - 1 \end{array}\right\} \quad \chi \to \chi.$$

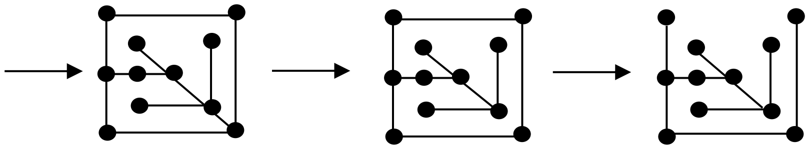
We continue this operation until there is only one face remaining. Applying the above operation we can convert any map to a tree. If there are *n* edges in this tree there are *n* + 1 vertices by Theorem 3 and so:

$$\chi = (n + 1) + 0 - n = 1.$$ ✋☺

Example 20:

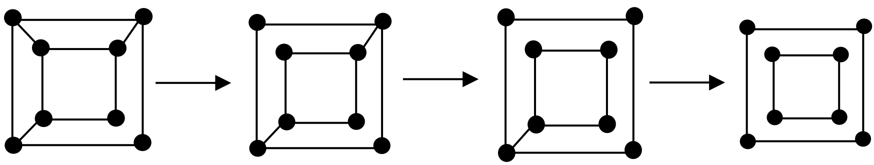


V = 11	V = 11	V = 11	V = 11
F = 6	F = 5	F = 4	F = 3



V = 11	V = 11	V = 11
F = 2	F = 1	F = 0

Example 21: Explain why the following example does not contradict Theorem 4.

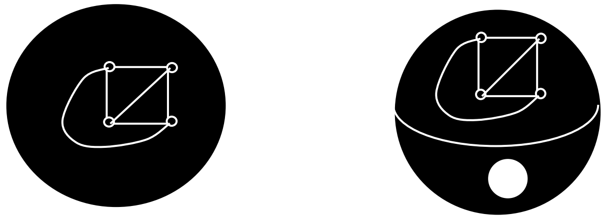


V = 8	V = 8	V = 8	V = 8
F = 4	F = 3	F = 2	F = 2
E = 11	E = 10	E = 9	E = 8
$\chi = 1$	$\chi = 1$	$\chi = 1$	$\chi = 2$

Solution: In deleting edges we must ensure that they are the boundary of separate faces. The last edge to be removed doesn't do this.

Since a disk can be cut out of a sphere, any planar graph can be drawn on a sphere without edges crossing.

On the other hand, if we can draw a graph on a sphere, without edges crossing, we can cut out a small hole in the middle of one of the faces and stretch the rest flat. In this way we will have a picture of the graph on a plane.



So a graph is planar if and only if it can be drawn, without crossings, on a sphere. This is useful because often a sphere is more convenient to work with.

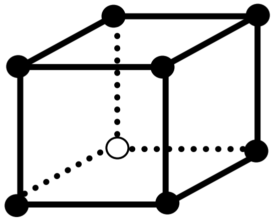
Theorem 4: (EULER'S FORMULA ON A SPHERE)
For any connected map drawn on a sphere:

$$V + F - E = 2.$$

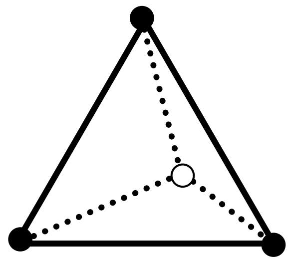
Proof: When a map is drawn on a sphere we have one extra face. ✋☺

A **polyhedron** (plural 'polyhedra') is a surface in 3 dimensions made up of planar faces, each of which is a polygon. These faces meet in straight edges and the edges meet in vertices. The network of vertices and edges is a graph and if the surface is expanded to a surface (imagine that the faces are made of some material that stretches and it is pumped up) we get a planar graph drawn on a sphere. So for polyhedra V + F – E = 2.

Example 22: Maps on a sphere:



V = 8
F = 6
E = 12
$\therefore\ \chi = 8 + 6 - 12 = 2.$



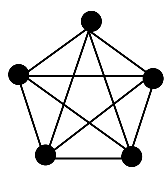
V = 4
F = 4
E = 6
$\therefore\ \chi = 5 + 5 - 8 = 2.$

§6.3. Proving That Certain Graphs Are Not Planar

To prove that a graph is planar we can simply draw it, with edges not crossing. But how do we show that a graph, such as, $K_{3,3}$ or K_5 is *not* planar? The technique discussed here is to work out the average number of edges per face and compare this to the smallest number of edges per face. But wait a minute. How can we count the faces unless we can draw it without edges crossing?

That's true, but the answer is to use Euler's formula for a sphere: V + F – E = 2

Example 23: K_5 is not planar.



Proof: For K_5 we have V = 5 and E = 10.
Suppose that K_5 is planar.
Then embedding it in a sphere we can deduce that the number of faces must be: F = 2 + E – V = 7.
The average number of edges per face must therefore be

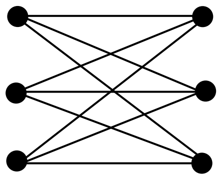
$$\frac{2E}{F} = \frac{20}{7} = 2\frac{6}{7}.$$

Why $\frac{2E}{F}$ and not just $\frac{E}{F}$? The reason is that every edge is associated with two faces – one on each side. So if you were to split each edge lengthwise, so that each half edge was associated with only one face, you'd have 2E half edges to share among the F faces.

Now we wanted to prove that K_5 can't be embedded in a sphere and we started out by supposing that it can be. We're clearly looking for a contradiction. So what's contradictory about the average number of edges per face being $2\frac{6}{7}$?

What is wrong is that it's less than 3. Every face must be surrounded by at least 3 edges. (A face bounded by 2 edges would require that the two edges connect the same two vertices, and a face bounded by just 1 edge would mean that the graph has a loop.) Now the average of a collection of numbers can't be less than the smallest of them. So here we have our contradiction!

Example 24: $K_{3,3}$ is not planar.



Proof: Here V = 6 and E = 9.
Suppose that $K_{3,3}$ can be embedded in a sphere.
The resulting map would have to have F faces where

$$6 + F - 9 = 2,$$

that is it must have 5 faces.
The average number of edges per face would therefore be

$$\frac{18}{5} = 3\frac{3}{5}.$$

That's not less than 3, so where's the contradiction? The contradiction is that it's less than 4. You see, in this graph there are no cycles of length 3. Each edge takes you from one set of vertices to the other. Going along another edge must take you back to a different vertex in the first set. The smallest cycles in this graph therefore have length 4. The boundary of a face must be a cycle in the graph. So the smallest number of edges for each face is 4. But the average of these numbers is less than 4. This can't be, and so we have our contradiction.

The **girth** of a graph is the length of the shortest cycle. The girth of K_5 is 3 but the girth of $K_{3,3}$ is 4. We get a contradiction if the average number of edges per face is less than the girth.

Theorem 5: If a connected graph has V vertices, E edges and a girth of g such that

$$\frac{2E}{2 + E - V} < g$$

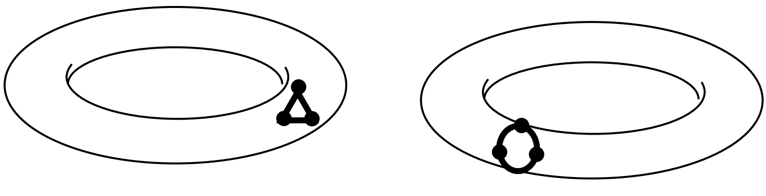
then the graph is not planar.

Proof: $\frac{2E}{2 + E - V}$ is the average number of edges per face, if the graph can be embedded in a sphere, with no eges crossing. If this is less than the length of the smallest cycle we have a contradiction.

§6.4. Graphs on a Torus

A **torus** is a doughnut shape, and because of the hole in the middle we have extra ability to embed graphs, without edges crossing.

Here are two different ways of embedding a triangle in a torus.



But to draw on an actual torus is difficult. However we can use a 2D model, as follows.

We draw the torus as a square, but a square with 'wraparound' both horizontally and vertically. So points on the right-hand edge are not really boundary points, because we identify those points with the corresponding point on the left-edge. If you move off to the right you disappear at the right edge and reappear at the left.

Points on the top edge are identified with the corresponding points on the bottom edge.